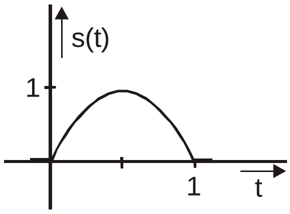
Musterlösung Aufgabe 1

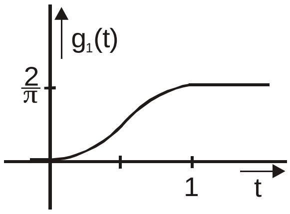
1.1 $s(t) = \sin(\pi t)\,\mathrm{rect}\left(t - \frac{1}{2}\right)$



$g_1(t) = s(t) * \varepsilon(t)$

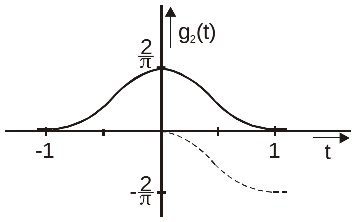
$g_1(t) = \int_{-\infty}^{t} s(\tau)\, d\tau = \int_{-\infty}^{t} \sin(\pi \tau)\,\mathrm{rect}\left(\tau - \frac{1}{2}\right) d\,\tau$

$g_1(t) = \frac{1}{\pi}\big(1 - \cos(\pi t)\big) \cdot \mathrm{rect}\left(t - \frac{1}{2}\right) + \frac{2}{\pi}\varepsilon(t-1)$



1.2 $g_2(t) = s(t) * h(t) = s(t) * \left[\varepsilon(t+1) - \varepsilon(t)\right]$

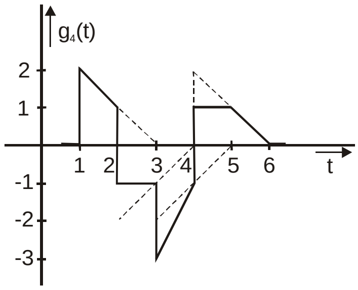
$= g_1(t+1) - g_1(t) = \frac{1}{\pi}\left[1 + \cos(\pi t)\right]\mathrm{rect}\left(\frac{t}{2}\right)$



1.3 $g_3(t) = s(t) * \delta'(t) = \frac{d}{dt} s(t) = \pi \cos(\pi t) \cdot \mathrm{rect}\left(t - \frac{1}{2}\right)$

1.4 $s_4(t) = \mathrm{rect}\left(t - \frac{3}{2}\right) = \varepsilon(t-1) - \varepsilon(t-2)$

$\Rightarrow \quad g_4(t) = h_\varepsilon(t-1) - h_\varepsilon(t-2)$



1.5 $h_1(t) = 0$ für $t < 0 \quad \Rightarrow$ kausal

$\int_0^{\infty} \left| h_1(\tau) \right| d\tau = \frac{1}{3} e^{-3} < \infty \quad \Rightarrow$ stabil

$h_2(t) \neq 0$ für $t < 1 \quad \Rightarrow$ nicht kausal

$\int_1^{-\infty} \left| h_2(\tau) \right| d\tau = -\frac{1}{3} e^{-3\tau} \Big|_1^{-\infty} = \infty \quad \Rightarrow$ nicht stabil

Musterlösung Aufgabe 2

2.1 $$c_k = \frac{2}{T}\int_0^{T/2} s_1(t)\cos(k\omega_0 t)\,dt = \frac{2}{T}\int_0^{T/2}\left(1-\frac{2}{T}t\right)\cos(k\omega_0 t)\,dt\,; \qquad \omega_0 = \frac{2\pi}{T}$$

$$c_k = \frac{2}{T}\int_0^{T/2}\cos(k\omega_0 t)\,dt - \frac{4}{T^2}\int_0^{T/2} t\cos(k\omega_0 t)\,dt = \frac{2}{T}\frac{1}{k\omega_0}\sin(k\omega_0 t)\Big|_0^{T/2} - \frac{4}{T^2 k\omega_0}\left[\frac{\cos(k\omega_0 t)}{k\omega_0} + t\cdot\sin(k\omega_0 t)\right]_0^{T/2}$$

$$c_k = \frac{2}{T}\cdot\frac{T}{k\cdot 2\pi}\underbrace{\left[\sin\left(k\frac{2\pi}{T}\frac{T}{2}\right)-\sin(0)\right]}_{=0} - \frac{4}{T^2}\frac{T}{k\cdot 2\pi}\left[\frac{T\cdot\cos\left(k\frac{2\pi}{T}\frac{T}{2}\right)}{k\cdot 2\pi} + \underbrace{\frac{T}{2}\sin\left(k\frac{2\pi}{T}\frac{T}{2}\right)}_{=0} - \frac{T\cdot\cos(0)}{k\cdot 2\pi}\right]$$

$$c_k = -\frac{2}{k\cdot\pi}\left[\frac{\cos(k\pi)}{k\cdot 2\pi} - \frac{1}{k\cdot 2\pi}\right] = \frac{1}{k^2\pi^2}\left[1-\underbrace{\cos(k\cdot\pi)}_{=(-1)^k}\right] = \begin{cases} 0 & k\neq 0 \quad \textit{gerade} \\ \dfrac{2}{k^2\pi^2} & k \quad \textit{ungerade}\end{cases}$$

$$c_0 = \frac{2}{T}\int_0^{T/2} s_1(t)\,dt = \frac{2}{T}\int_0^{T/2} dt - \frac{4}{T^2}\int_0^{T/2} t\,dt = 1-\frac{1}{2} = \frac{1}{2}$$

2.2 $${}^1c_k = c_k e^{-jk\omega_0\frac{T}{4}} = c_k\cdot e^{-jk\frac{\pi}{2}} = c_k\left[\cos\left(k\frac{\pi}{2}\right) - j\sin\left(k\frac{\pi}{2}\right)\right]$$ mit 2.1 $\Rightarrow$

$${}^1c_k = \begin{cases} \dfrac{1}{2} & k=0 \\ 0 & k = \textit{gerade} \\ -\dfrac{2}{k^2\pi^2}\,j\cdot\sin\left(k\dfrac{\pi}{2}\right) & k = \textit{ungerade}\end{cases}$$

$$\mathrm{Re}\left\{{}^1c_k\right\} = \begin{cases} \dfrac{1}{2} & \textit{für } k=0 \\ 0 & \textit{sonst}\end{cases} \qquad \mathrm{Im}\left\{{}^1c_k\right\} = \begin{cases} 0 & \textit{für } k = \textit{gerade} \\ -\dfrac{2}{k^2\pi^2}\sin\left(k\cdot\dfrac{\pi}{2}\right) & \textit{für } k = \textit{ungerade}\end{cases}$$

2.3 $$s_2(t) = \mathrm{rect}\left(\frac{t}{T/2}\right) * \frac{6}{T}\mathrm{rect}\left(\frac{t}{T/6}\right)$$

$$S_2(j\omega) = \frac{6}{T}\cdot\frac{T}{2}si\left(\frac{\omega T}{4}\right)\cdot\frac{T}{6}si\left(\frac{\omega T}{12}\right) = \frac{T}{2}si\left(\frac{\omega T}{4}\right)\cdot si\left(\frac{\omega T}{12}\right)$$

2.4 $$s_{2p}(t) = s_2(t) * \sum_{n=-\infty}^{\infty}\delta(t-nT)$$

$$S_{2p}(j\omega) = S_2(j\omega)\cdot\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi}{T}k\right)$$ mit $\omega_0 = \frac{2\pi}{T}$

$$S_{2p}(j\omega) = \pi\sum_{k=-\infty}^{\infty} si\left(k\cdot\frac{\pi}{2}\right) si\left(k\cdot\frac{\pi}{6}\right)\ \delta(\omega - k\omega_0)$$

mit $$S_{2p}(j\omega) = 2\pi\sum_{k=-\infty}^{\infty}{}^2c_k\ \delta(\omega-k\omega_0) \quad\Rightarrow\quad {}^2c_k = \frac{1}{2}si\left(k\cdot\frac{\pi}{2}\right) si\left(k\cdot\frac{\pi}{6}\right)$$

Musterlösung Aufgabe 3

3.1 $F(s) = 2\frac{s}{(s+(1+j))(s+(1-j))(s+2)} = 2\frac{s}{((s+1)^2+1)(s+2)}$

3.2 $F(s) = 2\left[\frac{a_1 s + a_2}{(s+1)^2+1} + \frac{a_3}{s+2}\right]$

$$= 2\frac{s^2(a_1+a_3)+s(2a_1+a_2+2a_3)+2(a_2+a_3)}{((s+1)^2+1)(s+2)}$$

$\Rightarrow \quad a_1 = -a_3 = a_2 \quad \Rightarrow \quad a_1 = a_2 = 1 = -a_3$

$$F(s) = 2\left[\frac{s+1}{(s+1)^2+1} - \frac{1}{s+2}\right]$$

3.3 mit Tabelle 5.1 und $\mathrm{Re}\{s\} > -1$

$$f(t) = 2\varepsilon(t)\left[e^{-t}\cos(t) - e^{-2t}\right]$$

3.4 $h(t) = \sum_{k=0}^{\infty} a^k \delta(t-kT)$ für $0 < a < 1$

$$H(s) = \sum_{k=0}^{\infty}\left(a\,e^{-sT}\right)^k = \frac{1}{1-a\,e^{-sT}}$$

3.5 keine Nullstellen

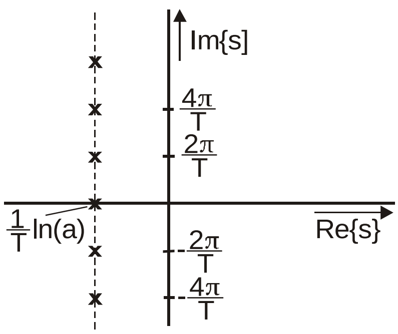
Polstellen: $1 - a\,e^{-sT} \overset{!}{=} 0 \quad \Rightarrow \quad \frac{1}{a} = e^{-\sigma T} e^{-j\omega T}$

$\Rightarrow \quad \sigma_P = \frac{1}{T}\ln(a)$ und $\omega_P T = \pm k \cdot 2\pi \qquad \omega_P = \pm k\frac{2\pi}{T}$

$$s_{pk} = \frac{1}{T}\ln(a) \pm j\,k\frac{2\pi}{T}$$

Konvergenzbereich: $\mathrm{Re}\{s\} > \frac{1}{T}\ln(a)$

da $h(t)$ kausal.



3.6 $j\omega$ – Achse im Konvergenzbereich $\Rightarrow H(j\omega)$ existiert

stabil, da alle Pole des <u>kausalen</u> Systems in der linken Halbebene der s – Ebene.

Musterlösung Aufgabe 4

4.1 $H(s) = H_0 \dfrac{a}{s+a}$ mit $\sigma = 0 \;\Rightarrow\; H(j\omega) = H_0 \dfrac{a}{j\omega + a} \;\Rightarrow\; H(j0) = H_0$

4.2 $h(t) = H_0 a\, e^{-at}\varepsilon(t) = H_0 \cdot a\, e^{-\frac{t}{T}}\varepsilon(t) \;\Rightarrow\; T = \dfrac{1}{a}$

4.3 $\left|\dfrac{H(j\omega_g)}{H(j0)}\right| = \left|\dfrac{a}{a + j\omega_g}\right| \overset{!}{=} \dfrac{1}{\sqrt{2}} \;\Rightarrow\; \dfrac{a^2}{a^2 + \omega_g^2} = \dfrac{1}{2} \;\Rightarrow\; \omega_g = a$

4.4 $H_1(s) = \dfrac{H(s)}{1 + k\, H(s)} = \dfrac{H_0 \cdot a}{a(1 + k\, H_0) + s}$

4.5 $H_1(j\omega) = \dfrac{H_0 \cdot a}{a(1 + k\, H_0) + j\omega} \;\Rightarrow\; H_1(j0) = \dfrac{H_0}{1 + k\, H_0}$

$h_1(t) = H_0 \cdot a\, e^{-a(1+kH_0)\,t}\varepsilon(t) \;\Rightarrow\; T_1 = \dfrac{1}{a(1 + k\, H_0)}$

$$\left|\frac{H_1(j\omega_{g_1})}{H_1(j0)}\right| = \left|\frac{a(1 + k\, H_0)}{a(1 + k\, H_0) + j\omega_{g_1}}\right| \overset{!}{=} \frac{1}{\sqrt{2}} \;\Rightarrow\; \frac{a^2 (1 + k\, H_0)^2}{\omega_{g_1}^2 + a^2 (1 + k\, H_0)^2} = \frac{1}{2} \;\Rightarrow \omega_{g_1} = a(1 + k\, H_0)$$

4.6 $\omega_{g_1} \overset{!}{=} 2\omega_g \;\Rightarrow\; a(1 + k\, H_0) \overset{!}{=} 2a \;\Rightarrow\; k = \dfrac{1}{H_0}$

$\Rightarrow T_1 = \dfrac{1}{2a}$; $H_1(j0) = \dfrac{H_0}{2}$

Musterlösung Aufgabe 5

5.1 $h_2(n) = h_1(n) * h_1(n) = \frac{1}{2}\left[\delta(n) + \delta(n-1)\right] * \frac{1}{2}\left[\delta(n) + \delta(n-1)\right]$

$$= \frac{1}{4}\left[\delta(n) + 2\,\delta(n-1) + \delta(n-2)\right]$$

5.2 $H_2(j\Omega) = \sum_{n=-\infty}^{\infty} h_2(n) e^{-jn\Omega} = \frac{1}{4}\left[1 + 2e^{-j\Omega} + e^{-j2\Omega}\right]$

$$= \frac{1}{4} e^{-j\Omega}\left[e^{j\Omega} + 2 + e^{-j\Omega}\right] = \frac{e^{-j\Omega}}{2}\left[1 + \cos(\Omega)\right]$$

$$\left|H_2(j\Omega)\right| = \frac{1 + \cos(\Omega)}{2}$$

$$\varphi_2(\Omega) = \arctan \frac{\mathrm{Im}\{H_2(j\Omega)\}}{\mathrm{Re}\{H_2(j\Omega)\}} = \arctan(-\tan(\Omega)) = -\Omega$$

5.3 $H_2(z) = \sum_{n=-\infty}^{\infty} h_2(n) z^{-n} = \frac{1}{4} + \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2}$

$\Rightarrow$ $H_2(z) = \frac{1}{4}\,\frac{1 + 2z + z^2}{z^2} = \frac{1}{4}\,\frac{(z+1)^2}{z^2}$

$z_{N1,2} = -1$; $z_{P1,2} = 0$ (zweifache Polstelle)

5.4 kausal: $h_2(n) = 0$ für $n < 0$

stabil: $\sum_n \left|h_2(n)\right| = 1 < \infty$

5.5 $H_3(z) = (z+1)^2 = z^2 + 2z + 1 = \frac{1}{z^{-2}}\left[1 + 2z^{-1} + z^{-2}\right]$

$\Rightarrow$ $h_3(n) = \delta(n+2) * \left[\delta(n) + 2\delta(n-1) + \delta(n-2)\right]$

$$h_3(n) = \delta(n+2) + 2\,\delta(n+1) + \delta(n)$$

5.6 $h_3(n) * h_4(n) \overset{!}{=} h_2(n)$

$\Rightarrow$ $h_4(n) = \frac{1}{4}\,\delta(n-2)$